Some Singularities of a Contact Transformation.

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§ 1. *Introduction.*

A number of papers have appeared in recent years on the singularities of point transformations. This paper discusses some of the singularities of a contact transformation.

The method of treatment in this paper will follow that of Urner in the *Transactions of the American Mathematical Society*, Vol. XIII, No. 2, pp. 232–264, April, 1912, on Certain Singularities of Point Transformations in Space of Three Dimensions.

In § 2 are given the preliminary hypotheses and notation. In § 3 some preliminary formulae are established. The discussion of the singularities divides itself into three cases, viz.: when the matrix of the Jacobian of the transformation is of rank two, one, or zero. These three cases are treated in §§ 4, 5, and 6, respectively. The irregularities which appear are in the order of contact of the transformed curves. In the latter part of § 5 is given an interesting geometric interpretation of the situation when the matrix of the Jacobian is of rank 1.

§ 2. *Preliminary Hypotheses and Notation.*

Let there be given a real contact transformation

$$x_1=X(x, y, p), \quad y_1=Y(x, y, p), \quad p_1=P(x, y, p). \tag{1}$$

We assume that X, Y, and P are single-valued and continuous when $(x, y, p) = (x_0, y_0, p_0)$ and that they possess with respect to these three variables partial derivatives of the first order also continuous there. The explicit use of any derivative of X, Y, or P shall imply its existence and continuity at the point under consideration, as well as the possession of these properties by all the other derivatives of X, Y, and P of the same and lower orders.

For a proper contact transformation, that is, one which is not a mere point transformation, not both X_p and Y_p can be zero. When we speak of a contact transformation we shall be understood to mean a proper contact transformation.

From the equations (1) we obtain

$$dy_1-p_1dx_1=\rho(x,\ y,\ p)\,(dy-pdx).$$

We shall assume that

$$\rho(x_0,\ y_0,\ p_0)=0,$$

but that

$$\rho(x,\ y,\ p)\not\equiv 0.$$

A lineal-element (x_0, y_0, p_0) for which $\rho=0$ is called a singular or critical lineal-element. It will presently appear that the vanishing of ρ is equivalent to the vanishing of the Jacobian J of the transformation.

We shall study the effect of the above transformation upon the curve, or more precisely the union of lineal-elements, given by

$$x=f(t),\quad y=g(t),\quad y'=g'(t)/f'(t)\equiv h(t),\tag{2}$$

where f and g are continuous, single-valued functions of the real variable t which we assume have continuous derivatives of orders $1, 2, \ldots, k+1$ at the point for which $t=t_0$. With these assumptions h is a single-valued function of the real variable t, continuous together with its derivatives of order $1, 2, \ldots, k$ at the point for which $t=t_0$, except for the zeros of $f'(t)$. We exclude from consideration those values of the parameter t for which $f'(t)=0$. Then we have certainly one of the quantities $f'(t_0)$, $g'(t_0)$, $h'(t_0)$ different from zero. The curve into which (2) is carried by the transformation is

$$\left.\begin{aligned} x_1&=X[f(t),\ g(t),\ h(t)]\equiv f_1(t),\\ y_1&=Y[f(t),\ g(t),\ h(t)]\equiv g_1(t),\\ p_1&=P[f(t),\ g(t),\ h(t)]\equiv h_1(t).\end{aligned}\right\}\tag{3}$$

Here f_1, g_1, and h_1 are single-valued and continuous and have continuous derivatives of orders $1, 2, \ldots, k$ for $t=t_0$ whenever f, g, and h are so endowed, provided X, Y, and P have continuous partial derivatives of orders $1, 2, \ldots, k$ at the point (x_0, y_0, p_0).

§ 3. *Preliminary Formulae.*

We shall need in the course of this paper to make use of some known relations between X, Y, and P when equations (1) represent a contact transformation. From Lie* we take the formulae

$$[XY]\equiv\begin{vmatrix} X_x, & X_y, & X_p\\ Y_x, & Y_y, & Y_p\\ -p, & 1, & 0\end{vmatrix}\equiv 0,\quad [PX]\equiv\begin{vmatrix} X_x, & X_y, & X_p\\ P_x, & P_y, & P_p\\ p, & -1, & 0\end{vmatrix}\equiv\rho,\quad [PY]\equiv\begin{vmatrix} Y_x, & Y_y, & Y_p\\ P_x, & P_y, & P_p\\ p, & -1, & 0\end{vmatrix}\equiv\rho P.$$

$$Y_x=PX_x-p\rho,\quad Y_y=PX_y+\rho,\quad Y_p=PX_p.\tag{4}$$

* "Berührungstransformationen," pp. 68 and 73.

From these relations it follows at once that J and ρ vanish simultaneously. Indeed

$$J \equiv \begin{vmatrix} X_x, & X_y, & X_p \\ Y_x, & Y_y, & Y_p \\ P_x, & P_y, & P_p \end{vmatrix} \equiv \rho[PX] \equiv \rho^2.$$

Let A_1, A_2, A_3; B_1, B_2, B_3; C_1, C_2, C_3 denote, respectively, the cofactors of the elements of the first, second, and third rows in the Jacobian J. Then J may be written in any one of the three following forms:

$$\left.\begin{array}{l} X_xA_1+X_yA_2+X_pA_3, \\ Y_xB_1+Y_yB_2+Y_pB_3, \\ P_xC_1+P_yC_2+P_yC_3. \end{array}\right\} \tag{5}$$

while

$$[XY] \equiv 0, \quad [PX] \equiv \rho, \quad [PY] \equiv \rho P \tag{6}$$

may be written in the respective forms

$$pC_1-C_2 \equiv 0, \quad pB_1-B_2 \equiv -\rho, \quad pA_1-A_2 \equiv \rho P. \tag{7}$$

From (5) and (7) we obtain when $J=0$

$$\begin{cases} (X_x+pX_y)A_1+X_pA_3=\rho PX_y, \\ (X_x+pX_y)A_2+pX_pA_3=-\rho PX_x, \end{cases}$$

$$\begin{cases} (Y_x+pY_y)B_1+Y_pB_3=-\rho Y_y, \\ (Y_x+pY_y)B_2+pY_pB_3=\rho Y_x, \end{cases}$$

$$\begin{cases} (P_x+pP_y)C_1+P_pC_3=0, \\ (P_x+pP_y)C_2+pP_pC_3=0. \end{cases}$$

From equations (4), (6), and (7) and the definitions of A_i, B_i, C_i we find:

$$\begin{aligned}
&B_1 \equiv X_pP_y-X_yP_p \equiv \rho_p, \\
&B_2 \equiv X_xP_p-X_pP_x=\rho+p\rho_p=\rho+pB_1, \\
&B_3 \equiv X_yP_x-X_xP_y=-\rho_x-p\rho_y, \\
&C_1 \equiv X_yY_p-X_pY_y=-\rho X_p, \\
&C_2 \equiv X_pY_x-X_xY_p=-p\rho X_p=pC_1, \\
&C_3 \equiv X_xY_y-X_yY_x=\rho(X_x+pX_y), \\
&A_1 \equiv Y_yP_p-Y_pP_y=\rho P_p-P\rho_p=\rho P_p-PB_1, \\
&A_2=p\rho P_p-P(\rho+p\rho_p)=p\rho P_p-PB_2, \\
&A_3=P(\rho_x+p\rho_y)-\rho(P_x+pP_y)=-\rho(P_x+pP_y)-PB_3.
\end{aligned}$$

When $\rho=0$ we obtain

$$C_1=C_2=C_3=0$$

$$\begin{array}{lll} B_1=\rho_p; & B_2=p\rho_p; & B_3=-(\rho_x+p\rho_y), \\ A_1=-PB_1; & A_2=-PB_2; & A_3=-PB_3. \end{array}$$

Whence

$$B_2=pB_1; \quad A_2=-pPB_1; \tag{8}$$
$$X_pB_3=-B_1(X_x+pX_y); \quad X_pA_3=PB_1(X_x+pX_y).$$

§ 4. *Matrix of Rank* 2.

We shall first discuss the singularities of the transformation (1) when the rank of the matrix of the determinant J is 2. From the equations of transformation we compute *

$$y_1'''=\frac{[PX]}{(X_x+X_yy'+X_{y'}y'')^3}y'''+\frac{U(x,y,y',y'')}{(X_x+X_yy'+X_{y'}y'')^3},$$

where the fraction whose numerator is $U(x, y, y', y'')$ stands for all of those terms which do not contain y'''. But

$$[PX]=\rho=0.$$

Then, in general, all curves through the point (x_0, y_0) and with a common value of y' and y'' will be transformed into curves having at the transformed point (x_1^0, y_1^0) a common value of y_1', y_1'', y_1'''. From the above formula we find

$$y_1^{(k)}=\frac{[PX]}{(X_x+X_yy'+X_{y'}y'')^3}y^{(k)}+U_k(x, y, y', \ldots, y^{(k-1)}),$$

where U_k stands for all of those terms which do not contain $y^{(k)}$. From this formula we see that, in general, all curves through the point (x_0, y_0) and with a common value for $y', y'', \ldots, y^{(k-1)}$ will be transformed into curves having at the transformed point (x_1^0, y_1^0) a common value for $y', y'', \ldots, y^{(k)}$. This gives us the

THEOREM: *If a contact transformation*

$$x_1=X(x, y, p), \quad y_1=Y(x, y, p), \quad p_1=P(x, y, p)$$

satisfies the conditions

a) X, Y, P *are of class* $C^{(k)}$ *at the point* P_0 *for which* $(x, y, p)=(x_0, y_0, p_0)$;

b) *The functional determinant* J *vanishes for*

$$x=x_0, \quad y=y_0, \quad z=z_0,.$$

but

$$J(x, y, p)\not\equiv 0;$$

c) *The matrix of the determinant* J *is of rank* 2, *then curves with contact of order* $k-1$ *at the point* P_0 *are transformed into curves with contact of order* k *at the transformed point* $P_0'(x_1^0, y_1^0)$, *i. e., the order of contact is increased by one.*

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* Lie, "Berührungstransformationen," p. 86.

It may happen, however, that the expression for y_1'',

$$y_1''=\frac{P_x+P_y y'+P_p y''}{X_x+X_y y'+X_p y''}$$

is indeterminate, whence of course $y_1^{(k)}$ becomes indeterminate for $k\geq 2$. In this case, in order to determine the curvature of the transformed curve, we proceed as follows. Let us state here that the slope of the transformed curves is always determined by the equations (1) of transformation. We have

$$dx:dy:dy'=f'(t_0):g'(t_0):h'(t_0),$$

and for the transformed curve

$$dx_1:dy_1:dy_1'=f_1'(t_0):g_1'(t_0):h_1'(t_0)$$

where

$$\left.\begin{aligned} f_1' &= X_x f'+X_y g'+X_p h', \\ g_1' &= Y_x f'+Y_y g'+Y_p h', \\ h_1' &= P_x f'+P_y g'+P_p h'. \end{aligned}\right\} \tag{9}$$

From these formulae we can determine the slope and radius of curvature of the transformed curve unless

$$f_1'(t_0)=g_1'(t_0)=h_1'(t_0)=0.$$

Since the matrix of the determinant of J is of rank 2, there exists one direction and radius of curvature for the original curve and only one for which the determination (9) fails. This is defined by the ratios

$$f':g':h'=I_1:I_2:I_3,$$

where I stands for that one of the letters A, B, C for which $I_1:I_2:I_3\neq 0:0:0$. We shall call this the *critical curvature* for the lineal-element (x_0, y_0, p_0).

Let us now assume that the curve (2) has the critical curvature. Suppose, however, that not all of the quantities $f_1''(t_0)$, $g_1''(t_0)$, $h_1''(t_0)$ are zero. Then

$$f_1'(t)\equiv f_1''[t_0+\theta_1(t-t_0)](t-t_0), \qquad (0<\theta_1<1)$$

with similar formulae for g_1', h_1'. Hence allowing t to approach t_0, we see that

$$dx_1:dy_1:dp_1=f_1''(t_0):g_1''(t_0):h_1''(t_0).$$

Placing $f'=kI_1$, $g'=kI_2$, $h'=kI_3$ we have

$$\left.\begin{aligned} f_1'' &= X_x f''+X_y g''+X_p h''+k^2U^2(X, I), \\ g_1'' &= Y_x f''+Y_y g''+Y_p h''+k^2U^2(Y, I), \\ h_1'' &= P_x f''+P_y g''+P_p h''+k^2U^2(P, I), \end{aligned}\right\} \tag{10}$$

where

$$U^2(FI)\equiv I_1^2F_{xx}+I_2^2F_{yy}+I_3^2F_{pp}+2I_2I_3F_{yp}+2I_3I_1F_{px}+2I_1I_2F_{xy}.$$

If we make a change of parameter, $t'=kt$, the functions f, g, h go over into

such functions $\bar{f}$, $\bar{g}$, $\bar{h}$ of t' that when $t'=kt_0$, $\bar{f}=I_1$, $\bar{g}=I_2$, $\bar{h}=I_3$. Then, let us drop dash and prime, merely supposing the necessary changes already made in the functions f, g, h as they stand. This done we have a right to set k in equations (10) equal to unity and $U^2(X, I)$, $U^2(Y, I)$, $U^2(P, I)$ become constants which we shall designate, respectively, by ρ_2, σ_2, τ_2. The curvature of the transformed curve is now determined by $f_1'' : h_1''$ unless

$$f_1'' : h_1'' = 0 : 0.$$

In this case the equations

$$\left.\begin{aligned} X_x f'' + X_y g'' + X_p h'' + \rho_2 = 0, \\ P_x f'' + P_y g'' + P_p h'' + \tau_2 = 0, \end{aligned}\right\} \tag{11}$$

are independent as equations for the determination of f'', g'', h''. At the point under consideration

$$I_1 = f', \ I_2 = g', \ I_3 = h' = \frac{f'g'' - g'f''}{f'^2} \tag{12}$$

We may now choose $(I_1, I_2, I_3 = B_1, B_2, B_3)$. By hypothesis $f' \neq 0$. Then $B_1 \neq 0$ and, hence, it follows from equations (8) that $B_2 \neq 0$. We are thus furnished with a third equation in f'', g'', h'':

$$B_2 f'' - B_1 g'' = -B_3 B_1^2. \tag{13}$$

The three equations (11) and (13) are independent and from them can be found the values of f'', g'', h'' for which the determination of the curvature of the transformed curve from equations (10) fails.

From the last of equations (12) by successive differentiations we obtain

$$h^{(j)} = \frac{f'g^{(j)} - g'f^{(j)}}{f'^2} + H_j, \tag{14}$$

where H_j depends upon the derivatives of f, g, of orders $1, 2, \ldots, j-1$. When $f' = B_1$ and $g' = B_2$ the equation (14) may be written in the form

$$B_2 f^{(j)} - B_1 g^{(j)} = B_1^2 (H_j - h^{(j-1)}). \tag{15}$$

The General Case. Let us now suppose that there exists a curve with the lineal-element (x_0, y_0, p_0) for which $f^{(j)}$, $h^{(j)}$ have such values that

$$f_1^{(j)}(t_0) = h_1^{(j)}(t_0) = 0,$$

for $j = 1, 2, \ldots, k-1$, but $[f_1^{(k)}]^2 + [h_1^{(k)}]^2 \neq 0$. Then let us join the point $t = t_0$ on the transformed curve with the point (x_1, y_1) for which $t = t_0 + \varepsilon$, by means of a secant. Then

$$x_1 - x_1^0 = f_1(t_0 + \varepsilon) - f_1(t_0) = \frac{\varepsilon^k}{k!} f_1^{(k)}[t_0 + \theta_1 \varepsilon], \qquad (0 < \theta_1 < 1),$$

with similar formulae for $y_1 - y_1^0$ and $p_1 - p_1^0$. Now, if we allow ε to approach

zero, we obtain for the determination of the curvature of the transformed curve

$$dx_1 : dy'_1 = dx_1 : dp_1 = f_1^{(k)}(t_0) : h_1^{(k)}(t_0).$$

If there exists a curve satisfying these requirements, then we shall say that (x_0, y_0, p_0) is a *singular lineal-element* of order $k-1$ at least. The order will be exactly $k-1$ if there is no curve for which

$$f_1^{(j)} = h_1^{(j)} = 0 \quad \text{for} \quad j = 1, 2, \ldots, k.$$

That is, the order of the singularity is precisely one unit less than the order of the highest derivatives which must be used to find the curvature of the transform of every curve with the lineal-element (x_0, y_0, p_0). Assuming for the moment that we have to do with a singular lineal-element of order $k-1$, let us designate as *critical curves* that family of curves which yield the values of $f^{(j)}$, $g^{(j)}$, $h^{(j)}$ necessary to make

$$f_1^{(j)} = h_1^{(j)} = 0 \quad \text{for} \quad j = 1, 2, \ldots, k-1.$$

By differentiating equations (9) $j-1$ times we get

$$\left.\begin{aligned} f_1^{(j)}(t) &= X_x f^{(j)}(t) + X_y g^{(j)}(t) + X_p h^{(j)}(t) + \rho_j, \\ g_1^{(j)}(t) &= \quad\quad\quad\quad \text{"} \quad\quad\quad\quad\quad + \sigma_j, \\ h_1^{(j)}(t) &= \quad\quad\quad\quad \text{"} \quad\quad\quad\quad\quad + \tau_j. \end{aligned}\right\} \tag{16}$$

The functions ρ_j, σ_j, τ_j depend on the derivatives of X, Y, P of orders $2, \ldots, j$ and on the derivatives of f, g, h of orders $1, 2, \ldots, j-1$. Then the critical curves have the curvature of their corresponding transformed curves determined by $f_1^{(k)}(t_0) : h_1^{(k)}(t_0)$. In computing these we must evaluate ρ_k, τ_k and these depend upon the derivatives of f, g, h of orders $1, 2, \ldots, k-1$, taken at $t = t_0$. The values of these derivatives are determined uniquely by the equations

$$\left.\begin{aligned} &f_1^{(j)} = \quad h_1^{(j)} = 0, \\ &B_2 f^{(j)} - B_1 g^{(j)} = B_1^2 (H_j - h^{(j-1)}), \qquad (j = 1, 2, \ldots, k-1). \end{aligned}\right\} \tag{15}$$

It is clear that all of the critical curves have with each other contact of order k at least. It is also clear from equations (16), if we put $j = k$, that the critical curves which have contact of order $k+1$ are transformed into curves with second order contact. Let us now assign arbitrary values to $f_1^{(k)}$, $g_1^{(k)}$, $h_1^{(k)}$, then the first and last of equations (16) and equations (15), all written for $j = k$, are three independent equations for the determination of $f^{(k)}$, $g^{(k)}$, $h^{(k)}$. We may now summarize our results as follows:

THEOREM: *All of those curves for which* $f^{(j)}$, $g^{(j)}$, $h^{(j)}$ *vanish, but for which* $[f^{(j+1)}]^2 + [g^{(j+1)}]^2 + [h^{(j+1)}]^2 \neq 0$ *are transformed into curves with a common*

direction and curvature. The curvatures of the transformed curves for different values of j are in general different.

§ 5. *Matrix of Rank* 1.

We will now consider the case in which the matrix of the Jacobian:

$$J\equiv\begin{vmatrix} X_x, & X_y, & X_p \\ Y_x, & Y_y, & Y_p \\ P_x, & P_y, & P_p \end{vmatrix}$$

of the transformation (1) is of rank 1. Let X_i, Y_i, P_i be a column of elements, not all of which vanish at (x_0, y_0, p_0), while i_x, i_y, i_p is a row possessing this property. The union of lineal-elements (2) is transformed by means of equations (1) into the union (3) and tangent curves go into tangent curves. If we desire to find the curvature of the transformed curve we find that it becomes indeterminate in form if, and only if, both f_1' and h_1' vanish simultaneously. Our problem is to evaluate this indeterminate form under the different circumstances which may arise.

In general, the direction and curvature of the transformed curve can be found from

$$dx_1 : dy_1 : dy_1' = f_1'(t_0) : g_1'(t_0) : h_1'(t_0),$$

where

$$\left.\begin{aligned} f_1' &= X_x f' + X_y g' + X_p h', \\ g_1' &= Y_x f' + Y_y g' + Y_p h', \\ h_1' &= P_x f' + P_y g' + P_p h'. \end{aligned}\right\} \tag{9}$$

The determination of these ratios will fail if, and only if, such a curve be taken that

$$X_x f' + X_y g' + X_p h' = 0. \tag{17}$$

All other curves then will be transformed into curves for which

$$dx_1 : dy_1 : dy_1' = X_i : Y_i : P_i,$$

and we may choose $X_i, Y_i, P_i = X_p, Y_p, P_p$. For $X_p \neq 0$, otherwise from the equation $Y_p - PX_p = 0$ we would have $Y_p = 0$ and the transformation would reduce to a mere point transformation which has been excluded. The equation (17) may be written in the form

$$X_p y'' + X_y y' + X_x = 0. \tag{18}$$

Since $X_p \neq 0$, this equation determines y'' and thus a definite radius of curvature is determined for certain plane curves with the common lineal-element (x_0, y_0, p_0). *If we avoid curves with the radius of curvature thus determined, and having the common lineal-element (x_0, y_0, p_0), then equations (9) determine for us the direction and curvature of the transformed curves.*

We are considering only those values of p for which $J=0$ and the matrix of J is of rank 1. Under these conditions we find $\rho_x+p\rho_y=0$. This yields a second homogeneous relation between f', g', h':

$$f'\rho_x+g'\rho_y=0. \tag{19}$$

From equations (17) and (19) we find

$$\begin{aligned} f' &= -\lambda X_p\rho_y \equiv \lambda\alpha, \\ g' &= \quad \lambda X_p\rho_x \equiv \lambda\beta, \\ h' &= \quad \lambda(X_x\rho_y-X_y\rho_x)\equiv\lambda\gamma, \end{aligned}$$

where α, β, γ are defined by these equations and λ is a factor of proportionality. The curvature of the transformed curve is then determined unless

$$f':g':h'=\alpha:\beta:\gamma.$$

Any curve for which $f':g':h'=\alpha:\beta:\gamma$ will be spoken of as a curve with the *critical curvature.* If a curve has this critical curvature, then, in general, the direction and curvature of the transformed curve can be found from f_1'', g_1'', h_1''. Placing $f'=\lambda\alpha$, $g'=\lambda\beta$, $h'=\lambda\gamma$, we have

$$\left.\begin{aligned} f_1'' &= X_xf''+X_yg''+X_ph''+\lambda^2U^2(X,\alpha,\beta,\gamma), \\ g_1'' &= Y_xf''+Y_yg''+Y_ph''+\lambda^2U^2(Y,\alpha,\beta,\gamma), \\ h_1'' &= P_xf''+P_yg''+P_ph''+\lambda^2U^2(Y,\alpha,\beta,\gamma), \end{aligned}\right\} \tag{20}$$

where

$$U^2(F,\alpha,\beta,\gamma)=\alpha^2F_{xx}+\beta^2F_{yy}+\gamma^2F_{pp}+2\beta\gamma F_{yp}+2\gamma\alpha F_{px}+2\alpha\beta F_{xy}.$$

If we make a change of parameter $t'=\lambda t$, the functions f, g, h go over into such functions f, $\bar{g}$, $\bar{h}$ of t' that when $t'=\lambda t_0$, $\bar{f}'=\alpha$, $\bar{g}'=\beta$, $\bar{h}'=\gamma$. Then let us drop dash and prime merely supposing the necessary changes already made in the functions f, g, h as they stand. This done we have a right to set λ in equations (20) equal to unity. Let us now put

$$\begin{aligned} \rho_2 &= U^2(X,\alpha,\beta,\gamma), \\ \sigma_2 &= U^2(Y,\alpha,\beta,\gamma), \\ \tau_2 &= U^2(P,\alpha,\beta,\gamma). \end{aligned}$$

We now consider ρ_2, σ_2, τ_2 as completely determined. The variable portions of the right members of (20) as f'', g'', h'' take on all possible values, preserve throughout, the ratios $X_i:Y_i:P_i$ so that

$$\begin{aligned} f_1'' &= \lambda X_i+\rho_2, \\ g_1'' &= \lambda Y_i+\sigma_2, \\ h_1'' &= \lambda P_i+\tau_2. \end{aligned}$$

Elimination of λ will give the following three equations:

$$Y_i(f_1''-\rho_2)-X_i(g_1''-\sigma_2)=0,$$
$$P_i(g_1''-\sigma_2)-Y_i(h_1''-\tau_2)=0,$$
$$X_i(h_1''-\tau_2)-P_i(f_1''-\rho_2)=0,$$

only two of which are independent. Between these two, the terms independent of f_1'', g_1'', h_1'' may be eliminated to yield a homogeneous linear relation. In particular, if $Y_i\neq 0$, this equation will be

$$(P_i\sigma_2-Y_i\tau_2)(Y_if_1''-X_ig_1'')-(Y_i\rho_2-X_i\sigma_2)(P_ig_1''-Y_ih_1'')=0. \tag{21}$$

It will be noticed that (21) is satisfied by putting

$$f_1'':g_1'':h_1''=X_i:Y_i:P_i, \tag{22}$$

whatever be the values of ρ_2, σ_2, τ_2.

The curvature of the transformed curve is now determined unless $f_1''=g_1''=h_1''=0$. We state this result as follows:

THEOREM: *All curves for which f', g', h', f'', g'', h'' have values which render $f_1'=g_1'=h_1'=0$, but for which not all f_1'', g_1'', h_1'' are zero are transformed into tangent curves at the transformed point (x_1^0, y_1^0) and all the transformed curves have the same radius of curvature at the transformed point.*

The vanishing of the three quantities

$$Y_i\rho_2-X_i\sigma_2, \quad P_i\sigma_2-Y_i\tau_2, \quad X_i\tau_2-P_i\rho_2$$

is the condition that the three equations

$$\left.\begin{aligned} X_xf''+X_yg''+X_ph''+\rho_2&=0,\\ Y_xf''+Y_yg''+Y_ph''+\sigma_2&=0,\\ P_xf''+P_yg''+P_ph''+\tau_2&=0, \end{aligned}\right\} \tag{23}$$

be consistent. If, then, we have

$$Y_i\rho_2-X_i\sigma_2=P_i\sigma_2-Y_i\tau_2=X_i\tau_2-P_i\rho_2=0,$$

there exist curves having the critical curvature, such that $f_1''=g_1''=h_1''=0$, and, therefore, our determination of curvature for the transformed curve is no longer valid.

Let us suppose that f'', g'', h'' have such values that equations (23) are satisfied. No two of these equations are independent as the matrix of J is now one. The curvature of the transformed curve will be given by $f_1''':g_1''':h_1'''$, where

$$\left.\begin{aligned} f_1'''&=X_xf'''+X_yg'''+X_ph'''+\rho_3,\\ g_1'''&=Y_xf'''+Y_yg'''+Y_ph'''+\sigma_3,\\ h_1'''&=P_xf'''+P_yg'''+P_ph'''+\tau_3, \end{aligned}\right\} \tag{24}$$

where

$$\begin{aligned}\rho_3 \equiv f''\,[&3X_{xx}f'+3X_{xy}g'+3X_{xp}h']\\ &+g''\,[Y_{yx}f'+Y_{yy}g'+Y_{yp}h'+2X_{yy}g'+2X_{yp}h'+2X_{xy}f']\\ &+h''\,[P_{xp}f'+P_{yp}g'+P_{pp}h'+2X_{pp}h'+2X_{yp}g'+2X_{xp}f']+\bar{\rho}_3,\end{aligned}$$

where $\bar{\rho}_3$ is a function of f', g', h', and third derivatives of X, Y, P, with similar expressions for σ_3, τ_3, viz., expressions which are *linear* in f'', g'', h''.

Equations (24) as f''', g''', h''', take on all possible values may be written in the form

$$\begin{aligned}f_1''' &= \lambda_3 X_i + \rho_3,\\ g_1''' &= \lambda_3 Y_i + \sigma_3,\\ h_1''' &= \lambda_3 P_i + \tau_3, \quad \lambda_3 \text{ being variable.}\end{aligned}$$

Elimination of λ_3 will give the following three equations:

$$\begin{aligned}Y_i(f_1'''-\rho_3)-X_i(g_1'''-\sigma_3)&=0,\\ P_i(g_1'''-\sigma_3)-Y_i(h_1'''-\tau_3)&=0,\\ X_i(h_1'''-\tau_3)-P_i(f_1'''-\rho_3)&=0,\end{aligned}$$

only two of which are independent. Between these two, the terms independent of f_1''', g_1''', h_1''', may be eliminated, to yield a homogeneous linear relation. In particular, if $Y_i \neq 0$, this equation will be

$$(P_i\sigma_3-Y_i\tau_3)\,(Y_if_1'''-X_ig_1''')-(Y_i\rho_3-X_i\sigma_3)\,(P_ig_1'''-Y_ih_1''')=0.$$

It will be noticed that this equation is satisfied by putting

$$f_1''' : g_1''' : h_1''' = X_i : Y_i : P_i \tag{25}$$

whatever be the values of ρ_3, σ_3, τ_3. If

$$Y_i\rho_3-X_i\sigma_3=P_i\sigma_3-Y_i\tau_3=X_i\tau_3-P_i\rho_3=0, \tag{26}$$

there exist curves having the given critical curvature such that $f_1'''=g_1'''=h_1'''=0$, and, therefore, our determination of curvature of the transformed curve from (24) is no longer valid. Equations (26) are *linear* equations in f'', g'', h'', and one of them is a consequence of the other two. There are two additional relations between f'', g'', h'':

$$X_xf''+X_yg''+X_ph''+\rho_2=0$$

and

$$B_2f''-B_1g''=-B_3B_1^2.$$

We have, thus, four independent equations for the determination of f'', g'', h''. In general, they cannot be satisfied. In case that these equations are satisfied f'', g'', h'' are uniquely determined and ρ_3, σ_3, τ_3 become certain fixed constants. The curvature of the transformed curve is now determined from $f_1^{(4)}$, $g_1^{(4)}$, $h_1^{(4)}$.

General Case. We now proceed to the discussion of the general case. Let us suppose that there exists a curve with the lineal-element (x_0, y_0, p_0) for which $f^{(j)}$, $g^{(j)}$, $h^{(j)}$ have such values that

$$f_1^{(j)}(t_0)=g_1^{(j)}(t_0)=h_1^{(j)}(t_0)=0, \qquad (j=1, \ldots, k-1),$$

but

$$[f_1^{(k)}]^2+[g_1^{(k)}]^2+[h_1^{(k)}]^2 \neq 0.$$

Then the direction of the transformed curve will be given by

$$\left.\begin{aligned} f_1^{(k)}&=X_x f^{(k)}(t)+X_y g^{(k)}(t)+X_p h^{(k)}(t)+\rho_k, \\ g_1^{(k)}&=Y_x f^{(k)}(t)+Y_y g^{(k)}(t)+Y_p h^{(k)}(t)+\sigma_k, \\ h_1^{(k)}&=P_x f^{(k)}(t)+P_y g^{(k)}(t)+P_p h^{(k)}(t)+\tau_k. \end{aligned}\right\} \tag{27}$$

The functions ρ_k, σ_k, τ_k, depend upon the derivatives of X, Y, P, of orders $2, \ldots, k$, and on the derivatives of f, g, h, of orders $1, 2, \ldots, k-1$, and are *linear* in $f^{(k-1)}, g^{(k-1)}, h^{(k-1)}$.

In computing $f_1^{(k)}(t_0)$, $g_1^{(k)}(t_0)$, $h_1^{(k)}(t_0)$, we must evaluate ρ_k, σ_k, τ_k, and these depend upon the derivatives of f, g, h, of orders $1, 2, \ldots, k-1$ taken at $t=t_0$. The values of these derivatives are uniquely determined by the equations

$$Y_i\rho_j-X_i\sigma_j=P_i\sigma_j-Y_i\tau_j=X_i\tau_j-P_i\rho_j=0,$$
$$f_1^{(j)}=g_1^{(j)}=h_1^{(j)}=0, \qquad (j=1, 2, \ldots, k-1).$$

Equations (27), as $f^{(k)}, g^{(k)}, h^{(k)}$ take on all possible values, may be written in the form

$$\begin{aligned} f_1^{(k)}&=\lambda_k X_i+\rho_k, \\ g_1^{(k)}&=\lambda_k Y_i+\sigma_k, \\ h_1^{(k)}&=\lambda_k P_i+\tau_k, \quad \lambda_k \text{ being variable.} \end{aligned}$$

Elimination of λ_k will give the following three equations:

$$\begin{aligned} Y_i(f_1^{(k)}-\rho_k)-X_i(g_1^{(k)}-\sigma_k)&=0, \\ P_i(g_1^{(k)}-\sigma_k)-Y_i(h_1^{(k)}-\tau_k)&=0, \\ X_i(h_1^{(k)}-\tau_k)-P_i(f_1^{(k)}-\rho_k)&=0, \end{aligned}$$

only two of which are independent. Between these two, the terms independent of $f_1^{(k)}, g_1^{(k)}, h_1^{(k)}$, may be eliminated, to yield a homogeneous linear relation. In particular, if $Y_i \neq 0$, this equation will be

$$(P_i\sigma_k-Y_i\tau_k)(Y_if_1^{(k)}-X_ig_1^{(k)})-(Y_i\rho_k-X_i\sigma_k)(P_ig_1^{(k)}-Y_ih_1^{(k)})=0.$$

It will be noticed that this equation is satisfied by putting

$$f_1^{(k)}:g_1^{(k)}:h_1^{(k)}=X_i:Y_i:P_i \tag{28}$$

whatever be the values of ρ_k, σ_k, τ_k.

An inspection of equations (22), (25) and (28) shows that the following theorem is true:

THEOREM: *Let there be given a lineal-element* (x_0, y_0, p_0) *for which the Jacobian of the transformation is of rank* 1. *Then, all curves, possessing the common lineal-element* (x_0, y_0, p_0) *will be transformed by means of equations* (1) *into curves, which at the transformed point* (x_1^0, y_1^0) *have a common tangent, and in addition a common radius of curvature given by*

$$dx_1 : dy_1 : dy_1' = X_i : Y_i : P_i .$$

In case the matrix of the Jacobian J, of the transformation is of rank 1, we find from the preliminary formulae

$$\rho(x, y, p) = 0, \quad \rho_p = 0, \quad \rho_x + p\rho_y = 0. \tag{29}$$

Now, $\rho(x, y, p) = 0$ is a differential equation.

Equations (29) *assure us that the critical lineal-element is one whose direction and point coincide with that of the tangent and point of tangency to the curve of the singular solution of the differential equation* $\rho = 0$.

Let us now examine more closely the equation

$$X_p y'' + X_y y' + X_x = 0. \tag{18}$$

We are talking about a particular lineal-element and hence y' is fixed. This equation determines y'', since $X_p \neq 0$. But if y' and y'' are fixed, then the radius of curvature is fixed. *We have then associated with each lineal-element of the singular solution curve a definite radius of curvature.* Denote by (α, β) the coordinates of the center of curvature and by (x, y) a point on the singular solution curve. The locus of the centers of critical curvature will be given by eliminating x, y, y', y'' from the following equations:

$$\rho(x, y, p) = 0, \quad \rho_p(x, y, p) = 0,$$
$$X_p y'' + X_y y' + X_x = 0,$$
$$\alpha = x - \frac{y'(1 + y'^2)}{y''}, \quad \beta = y + \frac{1 + y'^2}{y''}.$$

Illustration.

$$X = p,$$
$$Y = \tfrac{1}{2}(y - px)^2 - p^2(y - px),$$
$$P = -2p(y - px) - x(y - px - p^2).$$

$$J = \begin{vmatrix} 0 & 0 & 1 \\ -p(y - px - p^2), & y - px - p^2, & -x(y - px - p^2) - 2p(y - px) \\ 2p^2 + px - (y - px - p^2), & -2p - x, & x^2 + 6px - 2y \end{vmatrix} = (y - px - p^2)^2 = \rho^2$$

whence

$$\rho = y - px - p^2.$$

The singular solution of this equation is

$$x^2+4y=0.$$

The matrix of the Jacobian J is of rank 1 at any lineal-element given by

$$(x,\ -\tfrac{1}{4}x^2,\ -\tfrac{1}{2}x), \tag{30}$$

or, in particular, at $(2, -1, -1)$. For any lineal-element given by (30) we have

$$\rho=\rho_p=\rho_x+p\rho_y=0.$$

The determination of the radius of curvature of the transformed curves from equations (9) fails if, and only if,

$$X_p y''+X_y y'+X_x=0. \tag{18}$$

In the present instance this equation reduces to

$$y''=0.$$

This makes the radius of curvature of the original curves infinite. Then, the critical curves (C) which pass through any point P of the envelope curve (E) are those curves which have the tangent to (E) at P for an inflectional tangent.

§ 6. *Matrix of Rank Zero.*

This case is impossible for a proper contact transformation. For, if $X_p=0$, it follows from the preliminary formulae that $Y_p=0$ also, and the transformation reduces to a mere point transformation.